PREFACE

The purpose of this *Instructor's Guide* is to save you time while helping you to teach a nice, honest, interesting, student-centered course. For each section, there are suggested additions to your lecture that can supplement (but not replace) things like the factoring of $x^2 + 3x + 2$ or the simplification of $2 \sin x \cos x$. Lecturing is not your only option, of course. This guide provides group activities, ready for reproducing, that will allow your students to discover and explore the concepts of algebra. You may find that your classes become more "fun", but I assure you that this unfortunate by-product of an engaged student population can't always be avoided.

This guide was designed to be used with *Algebra and Trigonometry, Fourth Edition* as a source of both supplementary and complementary material. Depending on your preference, you can either occasionally glance through the *Guide* for content ideas and alternate approaches, or you can use it as a major component in planning your day-to-day classes. In addition to lecture materials and group activities, each section has examples, sample homework assignments, and reading quizzes.

For many students, college algebra is the class where they decide whether or not they are good at math, and whether or not they like it. Therefore, teaching college algebra and trigonometry is an important, noble task. It is my hope that this book will make that task a little easier.

For giving me this opportunity, I thank Jim Stewart, Lothar Redlin, Saleem Watson, Richard Stratton, and Bob Pirtle. Samantha Lugtu has been a wonderful editor, tolerant of various sillinesses that accompanied emailed chapters. Thanks also to Jacqueline Ilg for her help and enthusiasm. If you like the way this book looks, if you admire the clarity of the graphics and the smoothness of its design, join me in thanking Andy Bulman-Fleming, the best typesetter in the business, who can typeset a chapter, catch every mistake therein, and beat me at online Scrabble, all in a single afternoon..

This book is dedicated to my department head, Doug Mupasiri, who has always encouraged me to take my career as a University of Northern Iowa mathematics professor in whatever direction it needed to go.

Douglas Shaw

CONTENTS

HOW TO USE THE INSTRUCTOR'S GUIDE

This *Instructor's Guide* provides further information for each section of *Algebra and Trigonometry, Fourth Edition*. Here is what you'll see:

1. **Suggested Time and Emphasis** These suggestions assume that the class is fifty minutes long. They also advise whether or not the material is essential to the rest of the course.
2. **Points to Stress** Major concepts that should drive the discussion.
3. **Sample Questions** Some instructors have reported that they like to open or close class by presenting a single question, either as a quiz or to start a discussion. Two types are included:
   - **Text Question:** These questions are designed for students who have done the reading, but haven't yet seen the material in class. You can use them to help ensure that the students are reading the textbook carefully.
   - **Drill Question:** These questions are straightforward "right down the middle" questions for students who have tried, but not necessarily mastered, the material.
4. **In-Class Materials** Use these ideas to help create a classroom atmosphere of experimentation and inquiry.
5. **Examples** Routine examples with all the computations worked out, designed to save class preparation time.
6. **Group Work** These group activities have been tested in the classroom. Suggestions for implementation and answers are provided first, followed by photocopy-ready handouts on separate pages. This guide's main philosophy of group work is that there should be a solid introduction to each exercise ("What are we supposed to do?") and good closure before class is dismissed ("Why did we just do that?")
7. **Homework Problems** For each section, a set of essential **Core Exercises** (a bare minimum set of homework problems) is provided. Using this core set as a base, a **Sample Assignment** is suggested.

TIPS ON IN-CLASS GROUP WORK

This *Instructor's Guide* gives classroom-tested group work activities for every section of *Algebra and Trigonometry, Fourth Edition*. One reason for the popularity of in-class group work is that *it is effective*. When students are engaged in doing mathematics, and talking about mathematics with others, they tend to learn better and retain the material longer. Think back to your own career: didn't you learn a lot of mathematics when you began teaching it to other people? Many skeptics experiment by trying group work for one semester, and then they get hooked. Pick a group activity from the guide that you like, make some photocopies, and dive in!

1. **Mechanics** Books and seminars on in-class group work abound. I have conducted some of these seminars myself. What follows are some tips to give you a good start:

   (a) **Do it on the first day.**
   The sources agree on this one. If you want your students to believe that group work is an important part of the course, you have to start them on the first day. My rule of thumb is "at least three times the first week, and then at least once a week thereafter." I mention this first because it is the most important tip.

   (b) **Make them move.**
   Ideally, students should be eye-to-eye and knee-to-knee. If this isn't possible, do the best you can. But it is important to have them move. If your groups are randomly selected, then they will have to get up and sit in a different chair. If your groups are organized by where they are seated in the classroom, make them move their chairs so they face each other. There needs to be a "break" between sitting-and-writing mode and talking-to-colleagues mode.

   (c) **Use the ideal group size.**
   Research has shown that the ideal group size is three students, with four-student groups next. I like to use groups of four: if one of them is absent (physically or otherwise), the group still has three participating members.

   (d) **Fixed versus random groups.**
   There is a lot of disagreement here. Fixed groups allow each student to find her or his niche, and allow you to be thoughtful when you assign groups or reassign them after exams. Random groups allow students to have the experience of working with a variety of people. I believe the best thing to do is to try both methods, and see which works best for you and your students.

   (e) **Should students hand in their work?**
   The advantage of handing in group works is accountability. My philosophy is that I want the group work to have obvious, intrinsic benefit. I try to make the experience such that it is obvious to the student that they get a lot out of participating, so I don't need the threat of "I'm grading this" to get them to focus. I sometimes have the students hand in the group work, but only as a last resort.

2. **Closure** As stated above, I want my students to understand the value of working together actively in their groups. Once you win this battle, you will find that a lot of motivation and discipline problems go away. I've found the best way to ensure that the students understand why they've done an activity is to tell them. The students should leave the room having seen the solutions and knowing why they did that particular activity. You can have the students present answers or present them yourself, whatever suits your teaching style. Transparencies and document cameras can allow students to skip the step of recopying their work on a black- or whiteboard.

   Here is another way to think about closure: Once in a while, use a future homework problem out as a group work. When the students realize that participating fully in the group work helps them in the homework, they get a solid feeling about the whole process.
3. **Introduction** The most important part of a group activity, in my opinion, is closure. The second most important is the introduction. A big killer of group work is that awful time between you telling your students they can start, and the first move of pencil on paper—the "what on earth do we do now?" moment. A good introduction should be focused on getting them past that moment. You don't want to give too much away, but you also don't want to throw them into the deep end of the swimming pool. In some classes, you may have to do the first problem with them, but in others you may just have to say, "Go!" Experiment with your introductions, but never neglect them.
4. **Help when you are needed** Some group work methods involve giving absolutely no help when the students are working. Again, you will have to find what is best for you. If you give help too freely, the students have no incentive to talk to each other. If you are too stingy, the students can wind up frustrated. When a student asks me for help, I first ask the group what they think, and if it is clear they are all stuck at the same point, I give a hint.
5. **Make understanding a goal in itself** Convey to the students (again, directness is a virtue here) that their goal is not just to get the answer written down, but to ensure that every student in their group understands the answer. Their work is not done until they are sure that every one of their colleagues can leave the room knowing how to do the problem. You don't have to sell every single student on this idea for it to work.
6. **Bring it back when you can** Many of the group works in this guide foreshadow future material. When you are lecturing, try to make reference to past group works when it is appropriate. You will find that your students more easily recall a particular problem they discussed with their friends than a particular statement that you made during a lecture.

The above is just the tip of the iceberg. There are plenty of resources available, both online and in print. Don't be intimidated by the literature—start it on the first day of the next semester, and once you are into it, you may naturally want to read what other people have to say!

HOW TO IMPLEMENT THE DISCOVERY PROJECTS

One exciting yet intimidating aspect of teaching a course is projects. An extended assignment gives students the chance to take a focused problem or project and explore it in depth — making conjectures, discussing them, eventually drawing conclusions and writing them up in a clear, precise format. *Algebra and Trigonometry, Fourth Edition* has links to many Discovery Projects. They are excellent and well thought out, and should be explored if possible. Here are some tips on ensuring that your students have a successful experience.

Time Students should have two to three weeks to work on any extended out-of-class assignment. This is not because they will need all this time to complete it! But a fifteen-to-twenty-day deadline allows the students to be flexible in structuring their time wisely, and allows the instructors to apply fairly strict standards in grading the work.

Groups Students usually work in teams and are expected to have team meetings. The main problem students have in setting up these meetings is scheduling. Four randomly selected students may find it difficult or impossible to get together for more than an hour or so, which may not be sufficient. One way to help your students is to clearly specify a minimum number of meetings, and have one or all group members turn in summaries of what was accomplished at each meeting. A good first grouping may be by location.

As stated earlier, studies have shown that the optimal group size is three people, followed by four. I advocate groups of four whenever possible. That way, if someone doesn't show up to a team meeting, there are still three people there to discuss the problems.

Before the first project, students should discuss the different roles that are assumed in a team. Who will be responsible for keeping people informed of where and when they meet? Who will be responsible for making sure that the final copy of the report is all together when it is supposed to be? These types of jobs can be assigned within the team, or by the teacher at the outset.

Tell the students that you will be grading on both content and presentation. They should gear their work toward an audience that is bright, but not necessarily up-to-speed on this problem. For example, they can think of themselves as professional mathematicians writing for a manager, or as research assistants writing for a professor who is not necessarily a mathematician.

If the students are expected to put some effort into the project, it is important to let them know that some effort was put into the grading. Both form and content should be commented on, and recognition of good aspects of their work should be included along with criticism.

One way to help ensure cooperation is to let the students know that there will be an exam question based on the project. If every member of the group does well on that particular question, then they can all get a bonus, either on the exam or on the project grade.

Providing assistance Make sure that the students know when you are available to help them, and what kind of help you are willing to provide. Students may be required to hand in a rough draft ten days before the due date, to give them a little more structure and to make sure they have a solid week to write up the assignment.

Individual Accountability It is important that the students are individually accountable for the output of their group. Giving each student a different grade is a dangerous solution, because it does not necessarily encourage the students to discuss the material, and may actually discourage their working together. A better alternative might be to create a feedback form If the students are given a copy of the feedback form ahead of time, and they know that their future group placement will be based on what they do in their present group, then they are given an incentive to work hard. One surprising result is that when a group consists of students who were previously slackers, that group often does quite well. The exam question idea discussed earlier also gives individuals an incentive to keep up with their colleagues.

HOW TO USE THE REVIEW SECTIONS AND CHAPTER TESTS

Review sections for chapters of a textbook are often assigned to students the weekend before a test, but never graded. Students realize they won't be evaluated on this work and often skip the exercises in place of studying previous quizzes and glancing at old homework. A more useful activity for students is to use the review sections in *Algebra and Trigonometry, Fourth Edition* to discover their precise areas of difficulty. Implemented carefully, these are a useful resource for the students, particularly for helping them to retain the skills and concepts they've learned. To encourage more student usage, try the following alternatives:

1. Make notes of the types of exercises students have had difficulty with during the course. During a review session, assign students to work on similar exercises in the review sections and go over them at the end of class. Also assign exercises reminiscent of the ones you plan to have on the exam.

2. Use the review section problems to create a game. For instance, break students into groups and have a contest where the group that correctly answers the most randomly picked review questions "wins". One fun technique is to create a math "bingo" game. Give each group a 5×5 grid with answers to review problems. Randomly pick review problems, and write the questions on the board. Make sure that for a group to win, they must not only have the correct answers to the problems, but be able to give sound explanations as to how they got the answers.

3. A simple way to encourage students to look at the chapter tests is to use one of the problems, verbatim, as an exam question, making no secret of your intention to do so. It is important that students have an opportunity to get answers to any questions they have on the chapter tests before the exam is given; otherwise, this technique loses a great deal of its value.

P PREREQUISITES

P.1 MODELING THE REAL WORLD WITH ALGEBRA

▼ Suggested Time and Emphasis

$\frac{1}{2}$–1 class. Review material.

▼ Points to Stress

1. Going from a verbal description of a quantity to an algebraic model.

2. Using algebra to solve applied problems.

▼ Sample Questions

- **Text Question:** The textbook finds and uses several algebraic formulas. Give an example of one of the formulas you read about.
- **Drill Question:** Find a formula for the distance you travel in a car in terms of the car's speed and the number of hours traveled.
  Answer: distance $=$ (speed) $\times$ (number of hours traveled), where distance is measured in any unit of length and speed in measured in those same units of length per hour.

▼ In-Class Materials

- Implicit in this section is the concept of converting from specific examples to general algebraic formulas. For example, assume that a movie theater sells adult tickets for eight dollars, student tickets for six dollars, and child tickets for three dollars. Find a formula for the amount of revenue from one show. Start with some specific examples (such as 5 adults, 24 students, and 3 children, for a total of $\$193$) and move to the formula $P = 8a + 6s + 3c$. Then give them time to come up with a formula that will work for any choice of ticket prices. Try to guide them to the general formula $P = xa + ys + zc$. Perhaps use the formula to find the sales if five adult tickets were sold for $\$10$ each, twenty-four student tickets were sold for $\$8$ each, and three child tickets were sold for $\$6$ each. (Answer: $\$260$)
- Foreshadow a few properties in the next section by describing them in words and having the students attempt to represent them algebraically. For example, state that "If you are multiplying two numbers together, the order in which you do it doesn't matter," and see if the students can come up with the formula $ab = ba$.
- Derive or remind the students of the formula $d = rt$. Now pose the question: You have to travel 240 miles. How fast do you have to travel to make the trip in 4 hours? In 3 hours? In 2 hours? Discover that if there is a series of these questions, the easiest way to solve them is to create the formula $r = 240/t$. Now ask the following whimsical question: What if we wanted to make it in a half hour? Fifteen minutes? Ten minutes? Ask the student what happens to the rate as we make the time demands more and more unrealistic.

- Assume a student has received the following test scores: 82, 68, and 79. A common question is "What do I need to get on the fourth test to get a B?" Assuming the grade is based on the average of the four tests, have the students figure out the answer (if 80 is a B, then the answer is 91). Derive the formula $A = \dfrac{229+s}{4}$ and use it to show the devastating effect a 0 has on one's average. If homework is part of the students' grade, this is a lesson they should learn early in the semester!

▼ Example

A formula for the sum of the first n odd integers can be obtained from looking at a table of values:

$$\begin{aligned} 1 &= 1 \\ 1+3 &= 4 \\ 1+3+5 &= 9 \\ 1+3+5+7 &= 16 \\ \text{The sum of the first } n \text{ odd integers} &= n^2 \end{aligned}$$

The pattern is not the same thing as a proof, of course. (This formula is proved later in the text.)

▼ Group Work 1: The Hourly Rule of Thumb

The students should not only leave this activity having had practice translating real-world problems into algebraic formulas, but they should also be able to quickly convert hourly to annual salaries. Many of them will not realize that one can multiply by 2000 in one's head. Make it explicit — double the number, add the zeros. When the activity is over, the students should be able to go from 7 dollars per hour to $\$14{,}000$ per year (and back) quickly and easily.

Answers:

1. The first job, since it is worth $\$30{,}000$ per year **2.** $\$24{,}000$ per year **3.** $A = 2000P$ **4.** $P = \frac{1}{2000}A$

▼ Group Work 2: Does Speeding Save You Time?

In this activity students will come up with a formula that I have not seen in any textbook and which occasionally comes in handy. During the introduction, make sure they understand the distinction between the total amount of time a trip takes if you speed, and the amount of time you save by speeding. Hand out the parts one at a time, only giving a group the next part when you are convinced that they understand the previous one. Make sure to have a class discussion when all the students are finished with Part 1. Discuss how, for short trips, speeding doesn't really save a significant amount of time. Question 1 of the second part is not necessarily difficult, but it will be unfamiliar. By attempting it, the students should learn the process of creating an algebraic formula as a way of generalizing repetitive examples.

Answers:

Part 1

1. An hour and a half **2.** Approximately 0.21 hours (about 13 minutes) **3.** Five minutes
4. Approximately 0.71 minutes (about 43 seconds) **5.** Approximately 0.57 hours (about 34 minutes)

6. Since you are travelling for a longer time in the slow case, more time is saved. To contrast: In the case of going out for doughnuts, the travel time is so short that the time saved by speeding is negligible.

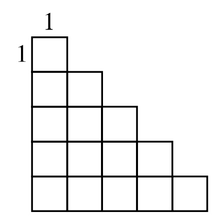
Part 2

1. $T = 1.5 - 90/(60 + S)$
2. The formula gives the right answer for $S = 10$ ($T = 0.21$). When $S = 0$ we should get zero, because if we do not speed, then no time is saved.
3. $T = \frac{D}{L} - \frac{D}{L+S}$. You may want to have your students simplify the formula: $T = \frac{DS}{L(L+S)}$. We can check our work by using the formula to solve the problems on the first part. Notice that $T = 0$ when $S = 0$, regardless of the values of D and L.

Part 3

1. 17.14 mi/h **2.** 3.53 mi/h **3.** $S = (TL^2)/(D - TL)$

▼ Group Work 3: Adding Them Up



This activity allows the students to attempt to find a useful, non-obvious algebraic formula. The proof that this formula is valid is given later in the text.

The pattern isn't immediately recognizable, but the students can eventually get it. It may help to ask the students to draw diagrams such as the one at right, and to think about areas.

The partial sums of this series are often called triangular numbers because they can be represented as above.

Ano ther way to think about the problem would be to consider the mean of the summands $\left(\frac{n+1}{2}\right)$, and realize that the sum will be the mean times the number of summands $\left(n \cdot \frac{n+1}{2}\right)$.

There are many ways to approach this problem. Try not to give too much guidance at first, besides the obvious hint that "Giving Up" is rarely a productive strategy!

Answers:

$$\begin{aligned} 1 &= 1 \\ 1+2 &= 3 \\ 1+2+3 &= 6 \\ 1+2+3+4 &= 10 \\ 1+2+3+4+5 &= 15 \\ 1+2+3+4+5+\cdots+n &= \frac{n(n+1)}{2} \end{aligned}$$

▼ Homework Problems

Core Exercises: 2, 12, 16, 23

Sample Assignment: 2, 3, 6, 12, 14, 16, 20, 23

GROUP WORK 1, SECTION P.1

The Hourly Rule of Thumb

1. You are looking for work and have two competing job offers. One of them offers you $15 per hour. The other offers you $23,000 per year. Assume for the moment that you will work 8 hours a day, 5 days a week, and have two weeks (unpaid) vacation per year. Which job is offering you more money?

2. The conversion between an hourly rate of pay and an annual rate of pay comes up quite often. When people discuss salaries, they often go back and forth. It is handy to find a formula that allows us to convert one to the other. Let's try another example. Assume you were offered 12 dollars per hour. How much would that be per year, assuming the same things we did in Problem 1?

3. Now come up with a formula that will find an annual salary, given an hourly rate of pay.

4. Sometimes the reverse is useful: You have an annual salary and you want to figure out the rate of pay per hour. Find a formula that will provide your hourly rate of pay.

5. Redo Problem 1, this time using your formula. Was it easier or harder?

GROUP WORK 2, SECTION P.1

Does Speeding Save You Time?

Part 1: The Warm-Up

1. You are traveling on a road where the speed limit is 60 miles per hour. Your destination is 90 miles away. How long does it take for you to get there, assuming that you travel at the speed limit?

2. Your friend riding shotgun, who often tries to get you to do bad things, urges you to exceed the speed limit by 10 miles per hour. Your other friend, who is relegated to the back seat, says that you don't really save any time by speeding. How much time would you save by going 70 miles per hour?

3. You arrive at your destination, and all is well. You are going to drive to the store to get doughnuts. It is a five-mile drive. You are going at 60 miles per hour. How long does it take you to arrive?

4. How much time would you save by going 10 miles per hour faster?

5. Now it is time to make the 90 mile trip back. There is construction! Oh no! The new speed limit is 35 miles per hour. Again, your Bad friend wants you to go 45 mi/h, and your Good friend wants you to go 35 mi/h. How much time do you save by speeding?

6. Why is it that speeding by 10 miles per hour gives you different time savings for the trip there and the trip back? Shouldn't the savings be the same?

Part 2: Generalizing the Parameters

When I was a student, my friend Ed noticed that when we did homework together, I would usually solve the first two problems much more slowly than he did, but then I would finish the entire assignment faster. The reason was that I would always try to work out how to do things in general, which takes longer than solving one particular problem as fast as you can, but then I could use my general methods on the rest of them. It is analogous to taking the time to learn the special features of a word-processing program. It takes you a long time to write that first letter, much longer than it would just typing it out, but then the rest of your letters go quickly.

Assume that the speed limit is 60 miles per hour, and the trip is going to be 90 miles, and you are going to speed. Let S be the amount (in miles per hour) by which you are going to exceed the speed limit.

1. Find a formula for the amount of time you save by going S miles per hour over the limit.

2. When generalizing a problem, it always pays to check your work. If we allow S to be ten, then we are back in the situation we had in the first problem. Does your formula give you the right answer for $S = 10$? What should the answer be if S is zero? Do you get that answer? What, physically, does it mean for S to be zero?

3. So now you have a formula that gives you time-savings for any amount of speeding. It is still not as useful as it could be. There is nothing special about a 90 mile trip. Let's let the trip length be D. Also, not all speed limits are 60 miles per hour. Let's let the speed limit be L. Now, find a formula that gives you the time-savings for a trip where the speed limit is L, the trip length is D, and you are speeding by S miles per hour over the limit. How can you check your work?

Part 3: Generalizing the Concept

You should be proud of your formula. You may even want to take it, and a calculator, with you on your next trip. The next thing that we should do is think, "Is this formula as useful as it can possibly be?" In one sense, it is—you want to know the effect of speeding, and it tells you. But in another sense, it is not—it does not tell you how fast to go.

Often, we want to get to our destination by a specific time. Here's a sample situation. We have a 90 mile trip to make, and the speed limit is 60 miles per hour. So that means it will take an hour and a half to get there. Unfortunately, we need to be at our destination in an hour and ten minutes. (I know I shouldn't advocate speeding. I'll tell you what—in the passenger seat you have a kidney that needs to get to the hospital so a little girl can live, and this very little girl will someday grow up to cure a major disease.)

1. By how much should you exceed the speed limit to save 20 minutes?

2. Assume that you had to be there in an hour and twenty five minutes. By how much should you exceed the speed limit to save 5 minutes?

3. So it looks like it would be handy to have a formula where you specify a trip distance D, a speed limit L, and a time savings T, and you find out by how much you need to speed to save T minutes. Try to find such a formula.

GROUP WORK 3, SECTION P.1

Adding Them Up

In this activity, we ask you to find a formula that is not obvious, but we have faith that you can do it.

1. We are going to be looking at sums of the following form: $1 + 2 + 3 + 4 + 5 + 6$. Our ultimate goal is to find a general formula for the sum of the first n numbers. First, we do a few examples. Fill out the following table:

$$\begin{aligned} 1 &= 1 \\ 1+2 &= 3 \\ 1+2+3 &= \\ 1+2+3+4 &= \\ 1+2+3+4+5 &= \end{aligned}$$

2. We want to find an algebraic formula for $1 + 2 + 3 + 4 + 5 + \cdots + n$. If we plug $n = 1$ into our formula, we want to get the answer 1. If we plug $n = 2$ into our formula, we want to get the answer 3, and so forth. Find such a formula. This will take a bit of thought.

P.2 THE REAL NUMBERS

▼ Suggested Time and Emphasis

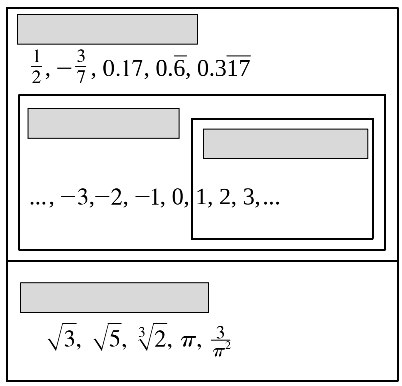
1–2 classes. Review material.

▼ Points to Stress

1. The various subsets of the real number line.
2. The algebraic properties of real numbers.
3. Sets and intervals, including the use of order symbols.
4. Absolute value.

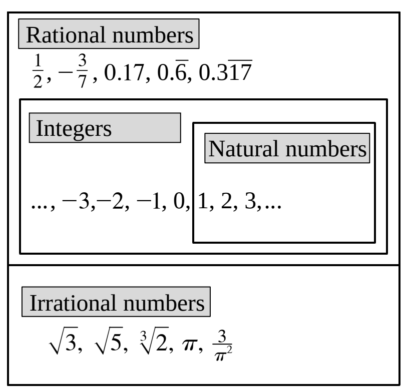
▼ Sample Questions

- **Text Question:** Consider the figure below from the text.



Fill in each gray box with a label from the following list: integers, irrational numbers, natural numbers, rational numbers.

Answer:



- **Drill Question:** (a) Find $(2, 5] \cup [3, 8)$. (b) Find $(2, 5] \cap [3, 8)$.

Answer: (a) $(2, 8)$ (b) $[3, 5]$

▼ In-Class Materials

- Point out that the set hierarchy in Figure 2 in the text isn't as simple as it may appear. For example, two irrational numbers can be added together to make a rational number ($p = 2 + \sqrt{3}$, $q = 2 - \sqrt{3}$) but two rational numbers, added together, are always rational (see Exercises 88 and 89). Similarly, mathematicians have long known that the numbers π and e ($\pi \approx 3.14159265\ldots$, $e \approx 2.718281828\ldots$) are irrational numbers, but it is unknown whether $\pi + e$ is rational or irrational. As people go farther in mathematics, they break the real numbers down into other types of sets such as the transcendentals, the algebraics, the normals, the computables, and so forth.

- The students probably already know how to multiply binomial expressions together; some of them already using the acronym FOIL to avoid thinking about the process altogether. Use the properties in this section to demonstrate why FOIL works:

$$\begin{aligned} (a+b)(c+d) &= a(c+d) + b(c+d) && \text{Distributive Property} \\ &= ac + ad + b(c+d) && \text{Distributive Property on the first term} \\ &= ac + ad + bc + bd && \text{Distributive Property on the second term} \end{aligned}$$

  If the example in the previous bullet item is used, then p and q can be multiplied together using FOIL to foreshadow the idea of conjugates.

- Warn the students that certain things cannot be rewritten. For example, an expression like $\dfrac{u}{u+1}$ cannot be simplified, while $\dfrac{u+1}{u}$ can also be expressed as $1 + \dfrac{1}{u}$. Similarly, there is no way to simplify $a + bc$. Another common pitfall can be pointed out by asking the question: "Is $-a$ a positive or negative number?" and having the students write their answer down before you go on to explain.

- Example 7(d) should be discussed: $|3 - \pi| = \pi - 3$. It touches on the ideas of distance, the absolute value of a negative number, and that we don't need to write out π to infinite precision to deduce that it is larger than three.

- Note that we never see intervals of the form $[3, \infty]$ or $(52, 3)$ and explain why they are not well-formed.

▼ Examples

- A nontrivial union of intervals: Let $S = \{x \mid x > 0, x \neq 1/n, \text{ where } n \text{ is a positive integer}\}$. This set is an infinite union of open intervals: $(1, \infty) \cup \left(\frac{1}{2}, 1\right) \cup \left(\frac{1}{3}, \frac{1}{2}\right) \cup \cdots$. Notice that each individual interval in the union is well-behaved and easy to understand. Also notice that even though S is an infinite union, the intersection of S with any positive open interval such as $(0.1, 5)$ becomes a finite union.

▼ Group Work 1: Foil the Happy Dolphin

The second page to this activity is optional, and should be given to a class that seems to have enjoyed working on the first page. If a group finishes early, ask them this surprisingly difficult follow-up question: "Write a paragraph explaining the answer to Problem 1 in such a way that a fourth-grader could understand it." The students may not succeed here, but the process of trying will help them to understand the concepts.

Answers:

1. It is easiest to explain by allowing the number to be x:

We start with	x
Adding 4 gives	$x+4$
Multiplying by 2 gives	$2(x+4)=2x+8$
Subtracting 6 gives	$2x+2$
Dividing by 2 gives	$\frac{1}{2}(2x+2)=x+1$
Subtracting the original number gives	$x+1-x=1$

2. Answers will vary.

3. Let a be the first three digits of a phone number and b be the last four. So the phone number has value $10^5 a+b$. Now we follow the steps:

1. We don't actually need a calculator here.

2. This is a.

3. $80a$

4. $80a+1$

5. $20{,}000a+250$

6. $20{,}000a+b+250$

7. $20{,}000a+2b+250$

8. $20{,}000a+2b$

9. $10{,}000a+b$, the original phone number.

▼ Group Work 2: A Strange Result

Note that $0.9999\overline{9}=1$. Not "is approximately equal to" but "equals." If your students do not believe it, ask them to find a number between $0.9999\overline{9}$ and 1.

Answers:

1. $\frac{1}{9}$ **2.** $\frac{4}{9}$ **3.** $\frac{5}{9}$ **4.** 1

▼ Group Work 3: What Are the Possibilities?

Many questions in this activity are not very challenging, but will show you the extent to which the class understands unions and intersections. The answers to the last few questions are actually very subtle, and while fun to think about, are not expected to be answered completely by the students. It is nice to establish early on that the students can be given questions to discuss in class which are worthwhile to think about even if they won't be on any exam. If a group finishes early and has some of the answers incorrect, point out that they need to fix something, without telling them which problem to fix.

Answers:

1. $(1,3)\cup(2,4)$ is one interval. $(1,3)\cup(4,5)$ is two intervals. It is not possible to choose values to make zero intervals or a single point.

2. $[1, 3] \cap [2, 4]$ is one interval. It is not possible to choose values to make two intervals. $[1, 3] \cap [4, 6]$ does not consist of any intervals. $[1, 3] \cap [3, 5]$ is the single number 3.

3. Yes

4. No. Choose $(0, \infty)$ and $(-\infty, 2)$.

5. No

6. No

7. No

8. Surprisingly, yes! Consider the union of all intervals of the form $\left[\frac{1}{n}, 2 - \frac{1}{n}\right]$ where n is a positive integer. The union of all such closed intervals is $(0, 2)$. Every number between zero and two is in infinitely many of the intervals, yet zero and two themselves are not in any of them.

▼ Group Work 4: Real-World Examples

Reinforce the distinction between closed and open intervals by trying to get the class to come up with real-world examples where each kind of interval is appropriate. Open the activity by referring to or copying the table above Example 5 from the text onto the board. Perhaps give them an example such as the following: "If the speed limit on a highway is 55 miles per hour and the minimum speed is 45 miles per hour, then the set of allowable speeds $[45, 55]$ is a closed interval." Gauge your class—don't let them start until they are clear as to what they will be trying to do. As the activity goes on, you can stop them halfway and let groups share one or two of their answers with the whole class to prevent groups from getting into mental ruts. Make sure to leave enough time for the students to discuss their answers. If you can foster an atmosphere of pride and kudos this early in the semester, it will make the group work easier later on.

Answers:

These will vary. Some typical answers can include the following:

- If a highway sign says, "SPEED LIMIT 70, MINIMUM 40," the set of allowable speeds is $[40, 70]$, a closed interval.
- If an apple-flavored product advertises that it contains real apple juice then the percentage of apple juice it contains is in $(0, 100]$, a half-open interval.
- The set of all possible temperatures (in $^\circ$C) that can be achieved in the universe is $[-273.16\ldots, \infty)$, a half-open infinite interval.

▼ Homework Problems

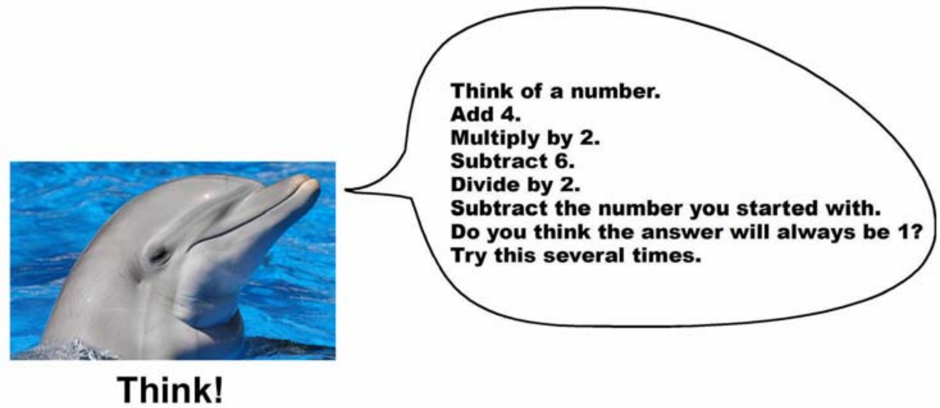
Core Exercises: 1, 7, 26, 38, 44, 60, 63, 70, 81, 85

Sample Assignment: 1, 5, 7, 12, 18, 22, 26, 32, 38, 40, 44, 46, 49, 57, 60, 63, 70, 76, 78, 81, 85, 93

GROUP WORK 1, SECTION P.2

Foil the Happy Dolphin

When I was in fourth grade, the following was written in a sidebar in my textbook, in a very friendly font next to a picture of a happy dolphin:



I tried it several times, and (can you believe it?) the answer was always 1. I was fascinated, because I had no idea why this was true. Later, when I learned about fractions, I tried it with $\frac{1}{2}$ and $\frac{2}{3}$ and it worked again. Much later, I learned about negative numbers and thought that surely this wondrous machine would break if I put in -2. But no, I still wound up with 1.

1. Does the Happy-Dolphin Procedure (HDP) always yield one? Show why it must be true, or find a number for which the HDP fails.

2. Come up with your own, more complicated, procedure that always gives the answer 1.

GROUP WORK 1, SECTION P.2

Foil the Happy Dolphin (Part 2)

Here is a chain email that has been going around:

```
From:  "Mia G. Nyuss" <mia@etaoinshrdlu.org>
To:  "Doug Shaw" <doug@shrdluetaoin.com>
Date:  Sun, 3 Apr 2016 11:55:14 -0600
Subject:  Wow, this is spooky

UNBELIEVABLE MATH PROBLEM
Here is a math trick so unbelievable that it is guaranteed to stump you.
(At least, it stumped me, and I have a degree in math from Yale.)
Personally, I would like to know who came up with this and where they found
the time to figure it out.  I still don't understand it!

1.  Grab a calculator.  (You won't be able to do this one in your head.)
2.  Key in the first three digits of your phone number (NOT the area code.)
3.  Multiply by 80.
4.  Add 1.
5.  Multiply by 250.
6.  Add the last 4 digits of your phone number.
7.  Add the last 4 digits of your phone number again.
8.  Subtract 250.
9.  Divide by 2.

Do you recognize this number?  Forward this email to twenty of your friends
and you get a magic wish.
```

3. What is the result of following the directions in the email? Prove that the result will be the same regardless of the phone number.

4. What is your magic wish?

GROUP WORK 2, SECTION P.2

A Strange Result

Express the following repeating decimals as rational numbers.

1. $0.1111\overline{1}$

2. $0.4444\overline{4}$

3. $0.5555\overline{5}$

4. $0.9999\overline{9}$

GROUP WORK 3, SECTION P.2

What Are the Possibilities?

1. Assume we have two intervals (a, b) and (c, d). Is it possible to choose a, b, c, and d so that $(a, b) \cup (c, d)$ can be expressed as a single interval? If so, how? Can it consist of two intervals? Zero intervals? Can it consist of a single point?

2. Assume we have two intervals $[a, b]$ and $[c, d]$. Is it possible to choose a, b, c, and d so that $[a, b] \cap [c, d]$ can be expressed as a single interval? If so, how? Can it consist of two intervals? Zero intervals? Can it consist of a single point?

3. Is the union of two infinite intervals always infinite?

4. Is the intersection of two infinite intervals always infinite?

5. Is it possible to take the union of two open intervals and get a closed interval? If so, how?

6. Is it possible to take the union of two closed intervals and get an open interval? If so, how?

7. Is it possible to take the union of infinitely many open intervals and get a closed interval? If so, how?

8. Is it possible to take the union of infinitely many closed intervals and get an open interval? If so, how?

GROUP WORK 4, SECTION P.2

Real-World Examples

Your textbook describes nine types of intervals. These are not just mathematical abstractions; there are real-world phenomena that represent each type of interval. Your task is to find quality examples of each type in the real world. For instance, if you have to be at least $5' \, 10''$ to ride a roller coaster, then the set of allowable heights in inches (according to the policy) is $[70, \infty)$, which is of the third type. (In practice, of course, a twenty-foot-tall person would be unable to ride the roller coaster, but the rule does not specifically prohibit it.)

1. $[a, b]$

2. $[a, b)$

3. $(a, b]$

4. (a, b)

5. (a, ∞)

6. $[a, \infty)$

7. $(-\infty, b)$

8. $(-\infty, b]$

9. $(-\infty, \infty)$

P.3 INTEGER EXPONENTS AND SCIENTIFIC NOTATION

▼ Suggested Time and Emphasis

$\frac{1}{2}$–1 class. Review material.

▼ Points to Stress

1. Definition of a^n in the cases where n is a positive or negative integer, or zero.

2. Algebraic properties of exponents.

3. Scientific notation and its relationship to significant digits.

▼ Sample Questions

- **Text Questions:**

  1. Why is it true that $a^m a^n = a^{m+n}$?

  2. Write $125{,}000{,}000$ in scientific notation.

  3. Why is scientific notation useful?

  Answers:

  1. $a^m a^n = \underbrace{(a \cdot a \cdot \cdots \cdot a)}_{m \text{ factors}} \underbrace{(a \cdot a \cdot \cdots \cdot a)}_{n \text{ factors}} = \underbrace{a \cdot a \cdot a \cdot \cdots \cdot a}_{m+n \text{ factors}} = a^{m+n}$

  2. $125{,}000{,}000 = 1.25 \times 10^8$

  3. It provides a compact way of writing very large numbers and very small numbers.

- **Drill Question:** Simplify $\left(\dfrac{a}{3}\right)^{-3} a^5$.

  Answer: $27a^2$

▼ In-Class Materials

- It is straightforward for the students to blindly memorize the rules of exponents. The problem is that they often forget the rules as quickly as they were memorized. One way to help retention is to assign each student a different rule to present to the class, emphasizing that the student has to explain *why* the rule is true. Make sure that the students have time to ask you (or each other) questions before the presentations.
- Negative exponents can be made to feel more intuitive with a table like this:

2^5	32
2^4	16
2^3	8
2^2	4
2^1	2

  Note that every step we are dividing by two, and continuing the pattern we get $2^0 = 1$ and $2^{-1} = \frac{1}{2}$.
- One of the early measurements of Mount Everest found it to be precisely $29{,}000$ feet tall. The surveyors were afraid that people would think it was an estimate—that they were merely saying that the mountain was between $28{,}500$ and $29{,}500$ feet tall. They were proud of their work. Discuss how they could have reported their results in such a way that people would know they were not just estimating. (Historically,

they lied and said it was $29{,}002$ ft. tall, so people would know there were five significant figures. Since the 1950s, the height has been listed as $29{,}029$ feet, although since 1999 some scientists believe a few feet should be added to that number.)

Answer: They could have reported it as being 2.9000×10^5 feet tall.

- When people talk about groups of objects, they often use the word "dozen" for convenience. It is easier to think about 3 dozen eggs than it is to think about 36 eggs. Chemists use a similar word for atoms: a "mole". There are a mole of atoms in 22.4 L of gas. (You can bring an empty 1 L bottle to help them picture it) and a mole of atoms in 18 mL of water. (Bring in a graduated cylinder to demonstrate.) A mole of atoms is 6.02×10^{23} atoms. Ask the students which is bigger:

  1. A mole or the number of inches along the Mississippi River
  2. A mole or a trillion
  3. A mole or the number of stars visible from Earth
  4. A mole or the number of grains of sand on Earth
  5. A mole or the number of sun-like stars in the universe

  The answer for all five is "a mole".

▼ Examples

- A product whose answer will be given by a calculator in scientific notation:

$$(4{,}678{,}200{,}000)\,(6{,}006{,}200{,}000) = 28{,}098{,}204{,}840{,}000{,}000{,}000$$

- A product whose answer contains too many significant figures for many calculators to compute accurately:

$$(415{,}210{,}709)\,(519{,}080{,}123) = 215{,}527{,}625{,}898{,}637{,}207$$

▼ Group Work 1: Water Bears

This is a good time to teach students how to find numbers such as $\sqrt[5]{12}$ on their calculators

Answers:

1. a
2. $\sqrt[5]{12} \approx 1.64375$
3. $a = 3$, $B = 40$
4. Answers will vary, given the fuzziness of real-world data. B is approximately 38 and a is approximately 2.5.

▼ Group Work 2: Guess the Exponent

This fun exercise will give students a feel for orders of magnitude. Put them into groups of three or four, and have them discuss the questions. Some are easy, some are tricky. When the groups seem to have achieved consensus, scramble them up, trying to make sure everybody is with two or three new people. Have the new groups try to achieve agreement. If they disagree, allow the groups to vote. For each problem, poll the groups, and then give the answer.

Answers: **1.** 2 **2.** 8 **3.** 3 **4.** 6 **5.** 0 **6.** 7 **7.** 2 **8.** 1 **9.** 3 **10.** 1 **11.** 5
12. 7 **13.** 8 **14.** 21 **15.** 5 **16.** 10 **17.** 5 **18.** 4 **19.** 2 **20.** 4 **21.** 3 **22.** 3

23. The answer, my friend, is blowing in the wind. Forty-two is an acceptable answer, as is "seven" if written in confident handwriting.

▼ Homework Problems

Core Exercises: 7, 11, 27, 36, 48, 51

Sample Assignment: 2, 7, 10, 11, 14, 21, 27, 31, 36, 40, 43, 48, 51, 55

GROUP WORK 1, SECTION P.3

Water Bears

The Warm-Up

1. Simplify the expression $\dfrac{Ba^{n+1}}{Ba^n}$.

2. If we know that $a^5 = 12$, find a.

Many types of populations grow this way: They start out growing slowly, then more quickly, then very, very quickly. For example, assume we put some tardigrades (cute microscopic organisms also known as "water bears") in a petri dish. Here is a table showing their population over time:

Day	Population
1	120
2	360
3	1080
4	3240
5	9720
6	29,160
7	87,480
8	262,440
9	787,320
10	2,361,960

The Water Bears

3. This type of growth is called **exponential growth**. After n days, there are Ba^n water bears. B and a are constants that we have to figure out. (Oh, by the way: in this context the word "we" actually means "you".) Find B and a.

4. The previous table wasn't really accurate. A true table of exponential growth might look more like this:

Day	**Population**
1	95
2	238
3	594
4	1484
5	3711
6	9277
7	23,190
8	57,980
9	144,960
10	362,390

Now calculate the real values of B and a.

The Hint Sheet

1. Calculate this ratio: the water bear population on Day 3 over the population on Day 2.

2. Now calculate the ratio of the population on Day 4 over the population on Day 3.

3. One more: calculate the ratio of the population on Day 5 over the population on Day 4.

4. How about in general? What is the ratio of the population on Day $n+1$ over the population on Day n?

5. Recall that the population at day n is given by Ba^n. (This is the definition of exponential growth.) Write an algebraic formula describing water bear populations. The left-hand side of the formula will be Ba^n.

6. Use the above to find a, and then find B.

GROUP WORK 2, SECTION P.3

Guess the Exponent

Most math problems ask you to figure out if the answer is 4, 5 or 6. Not these! We are going to give you the questions *and* the answers. For your comfort and convenience, the answers will be written in scientific notation. The only thing you have to do is supply the exponent. How hard can that be? How hard, indeed. . .

1. Approximately how many countries are there in the world?

   Answer: $2.0 \times 10^{\square}$

2. What was the total number of credit card accounts in the United States in 2013?

   Answer: $4 \times 10^{\square}$

3. How many crochet stitches are there in an adult-sized mitten?

   Answer: $1.7 \times 10^{\square}$

4. How many people lived in the United States in 1800?

   Answer: $5 \times 10^{\square}$

5. How many president's faces are on Mount Rushmore?

   Answer: $4 \times 10^{\square}$

6. Approximately how many species of insect are there in the world?

   Answer: $3 \times 10^{\square}$

7. How many movies were released in the United States in 2013?

   Answer: $1.5 \times 10^{\square}$

8. How many breeds of domestic cat are there?

   Answer: $4.6 \times 10^{\square}$

9. What is the longest a Redwood tree can live?

   Answer: $2 \times 10^{\square}$ years

10. How many playing cards are there in a hand of bridge?

    Answer: $1.3 \times 10^{\square}$

11. The average human head has how many hairs?

    Answer: Between $0.8 \times 10^{\square}$ and $1.2 \times 10^{\square}$

12. How many books are there in the United States Library of Congress?

    Answer: $1.8 \times 10^{\square}$

13. How many documents are there in the United States Library of Congress (including books, manuscripts, recordings, photographs, etc.)?

Answer: $1.2 \times 10^{\boxed{}}$

14. How many sun-like stars (stars that have planets) are there in the universe?

Answer: $1 \times 10^{\boxed{}}$

15. How far is it from the earth to the moon (on average)?

Answer: $2.5 \times 10^{\boxed{}}$ miles

16. Approximately how many neurons are there in a human brain?

Answer: $1 \times 10^{\boxed{}}$

17. How many words are there in the English language?

Answer: $8.16 \times 10^{\boxed{}}$

18. How many words are there in a typical novel?

Answer: Between $4.5 \times 10^{\boxed{}}$ and $15 \times 10^{\boxed{}}$

19. How many paintings did the artist Van Gogh create?

Answer: $8 \times 10^{\boxed{}}$

20. How many feet deep is the deepest spot in the ocean (the Mariana Trench in the Pacific Ocean)?

Answer: $3.6 \times 10^{\boxed{}}$

21. How many miles long is the Mississippi River?

Answer: $2.35 \times 10^{\boxed{}}$

22. How many miles long is the Nile River?

Answer: $4.15 \times 10^{\boxed{}}$

23. How many seas must the white dove sail before she sleeps in the sand?

Answer:

P.4 RATIONAL EXPONENTS AND RADICALS

▼ Suggested Time and Emphasis

$\frac{1}{2}$ –1 class. Review material.

▼ Points to Stress

1. Definition of a^n in the cases where n is a rational number.
2. Algebraic simplification of expressions involving rational exponents and radicals.
3. Rationalizing denominators.

▼ Sample Questions

- **Text Question:** What is the definition of $a^{3/4}$?
  Answer: $a^{3/4} = \sqrt[4]{a^3}$
- **Drill Question:** Simplify $\dfrac{\sqrt[3]{a^2}}{\sqrt[3]{a}} a^5$.
  Answer: $a^{16/3}$

▼ In-Class Materials

- Write "$\sqrt{ab} = \sqrt{a}\sqrt{b}$" on the board, and ask the students if this statement is always true. Point out (or have them discover) that this is a false statement when a and b are negative.
- There is a cube root magic trick that illustrates an interesting property of the cube function. A student takes a two-digit integer, cubes it, and gives you the answer (for example, 274,625). You knit your brow, think, and give the cube root (65) using the power of your mind. If the students are not impressed, stop there. If they are, do it a second time, getting the answer wrong by 1. Try it a third time, and again get the correct answer. Have them try to figure the trick out, and promise to reveal the secret if they do well on a future quiz or test.

  The trick is as follows. Memorize the first ten cubes:

x	0	1	2	3	4	5	6	7	8	9
x^3	0	1	8	27	64	125	216	343	612	729

  Notice that the last digit of each cube is unique, and that they aren't well scrambled (2's cube ends in 8 and vice versa; 3's ends in 7 and vice versa). So the last digit of the cube root is easily obtained (274,625 ends in 5, so the number you were given is of the form ?5). Now the first digit can be obtained by looking at the first three digits of the cube, and seeing in which range they are. (274 is between 6^3 and 7^3, so the number you were given is 65). Make sure to tell the students that if they do this trick for other people, they should not get the answer too fast, and they should get it wrong in the last digit occasionally. This will make it seem like they really can do cube roots in their heads.
- After doing some standard examples of rationalizing the denominator to put a fraction in standard form, try doing a few examples involving multiplying by a conjugate such as

$$\frac{5}{2+\sqrt{3}} = \frac{5}{2+\sqrt{3}}\left(\frac{2-\sqrt{3}}{2-\sqrt{3}}\right) = \frac{10-5\sqrt{3}}{4-9} = -2-\sqrt{3}$$

▼ Example

A fractional exponent simplification that works out nicely: $\dfrac{\sqrt[3]{8^5}}{\sqrt[3]{8^4}} = \sqrt[3]{\dfrac{8^5}{8^4}} = \sqrt[3]{8} = 2.$

▼ Group Work 1: Find the Error

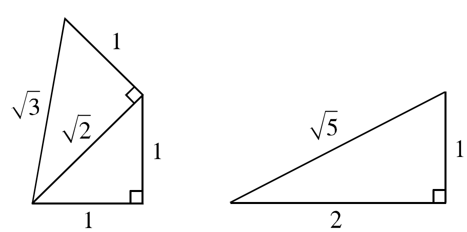
The student is invited to pick a hole in someone else's reasoning. Many students do not understand, at first, what they are trying to do—they will tend to point at the conclusion and say words to the effect of "$-1 \neq 1$, so that is the error," or they will start at a given step, proceed differently, and not wind up with a paradox. They will have to be guided to the idea that they must actually analyze *someone else's* reasoning, and find where that other person made the mistake.

Answer: The definition $a^{m/n} = \sqrt[n]{a^m}$ is valid only if m/n is in lowest terms.

▼ Group Work 2: Constructing $\sqrt{2}$

This activity could be expanded by a discussion of straight-edge and compass constructions, but the main idea is to remind the students of the Pythagorean Theorem. Introduce the activity by showing how it is easy (using a ruler) to make a line segment that is 1 inch, 2 inches, or 3 inches long. Hand out rulers and protractors, and have them try to make a segment that is $\sqrt{2}$ inches long. If they are stuck, suggest that they make a right triangle, and remind them of the Pythagorean Theorem. If a group finishes early, ask them to try and get a segment $\sqrt{5}$ units long, and then $\sqrt{3}$ units long. This can be made into an out-of-class project by asking the students to research and discuss the Wheel of Theodorus, which is a lovely extension of the concepts addressed in this activity.

Answer



▼ Homework Problems

Core Exercises: 4, 10, 24, 36, 45, 53, 62, 74, 90

Sample Assignment: 4, 8, 10, 14, 21, 24, 26, 30, 36, 39, 45, 50, 53, 57, 60, 62, 67, 74, 82, 90, 95

GROUP WORK 1, SECTION P.4

Find the Error

It is a beautiful Autumn day. Everyone around you is happy and excited because school has begun, and it is time to begin the joy of learning and hard work as opposed to the long summer of idle hooliganism. You are particularly happy because you are taking College Algebra. You sit at a picnic table, set up your thermos of cold milk and a peanut-butter and banana sandwich, and start reading where you had left off. Suddenly, you are aware of an odd odor, and turn around to see a wild-eyed ten-year old boy licking a giant lollipop. “What are you reading?” he asks.

“I am reading *College Algebra, Seventh Edition*,” you say. “It is a bit advanced for you, but it is jam-packed with useful and important information. Worry not, lad, the day will come when you, too, are able to read this wonderful book.”

“I’ve already read it,” the boy says smugly, “and think it is full of LIES.”

“What do you mean?” you ask incredulously. “James Stewart, Lothar Redlin and Saleem Watson have taken some of the greatest knowledge of our civilization, melted it down, mixed it with love, and put it into my textbook.”

“Great knowledge, huh?” he asks. “Tell me, what is $(-1)^1$?”

“Why, that is equal to -1,” you answer. “Any number to the first power is itself. I knew that even before taking College Algebra!”

“Okay then, what is $(-1)^{6/6}$?”

“Well, by the definition of rational roots, that would be $\sqrt[6]{(-1)^6} = \sqrt[6]{1} = 1$.”

“I thought you said $(-1)^1 = -1$, and now you are saying that $(-1)^{6/6} = 1\ldots$ In other words, you are saying that $-1 = 1$!”

You can’t be saying that, can you? The rude stranger must have made a mistake!

What was his mistake? Find the error.

GROUP WORK 2, SECTION P.4
Constructing $\sqrt{2}$

It is possible to use your ruler to make a line segment exactly 2 inches long. Your challenge is to make one that is exactly $\sqrt{2}$ inches long.

P.5 ALGEBRAIC EXPRESSIONS

▼ Suggested Time and Emphasis

$\frac{1}{2}$ – 1 class. Review material.

▼ Points to Stress

1. Definition of and algebraic operations with polynomials.
2. Special product formulas.

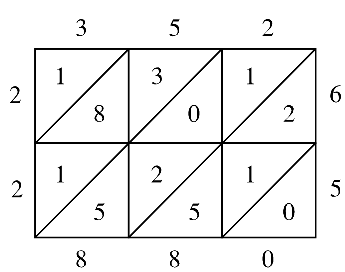
▼ Sample Questions

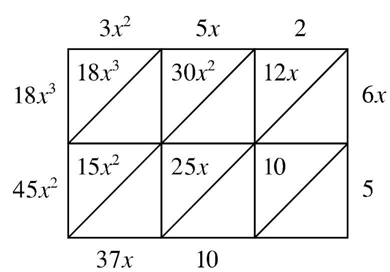
- **Text Question:**
  (a) Give an example of an expression that is a polynomial.
  (b) Give an example of an expression that is not a polynomial.
- **Drill Question:** Compute $\left(x^2 - x + 2\right)(x - 5)$.
  Answer: $x^3 - 6x^2 + 7x - 10$

▼ In-Class Materials

- Some students have learned the "standard algorithm" (long multiplication) for multiplying two numbers together, while others may have learned different algorithms, such as the "grid method" or "lattice multiplication." The table-form algorithm given in the text is analogous to long multiplication. Multiply 352 by 65 using long multiplication, and then multiply $3x^2 + 5x + 2$ by $6x + 5$ using the algorithm in the text. If your students are used to "lattice multiplication" this can also be done. One doesn't really need the "lattice" but the process can be put into the same form they are used to.

$$\begin{array}{r} 352 \\ \times\ 65 \\ \hline 1760 \\ 2112 \\ \hline 22880 \end{array} \qquad \begin{array}{rrrr} & 3x^2 + & 5x + & 2 \\ \times & & 6x + & 5 \\ \hline & 15x^2 + & 25x + & 10 \\ 18x^3 + & 30x^2 + & 12x & \\ \hline 18x^3 + & 45x^2 + & 37x + & 10 \end{array}$$





- In chemistry, the pressure, volume, quantity and temperature of a gas are related by the equation

$$PV = nRT$$

where P is pressure, V is volume, n is quantity (in moles), and T is temperature. (R is a constant called the ideal gas constant.) Present the formula this way:

$$P = \frac{nRT}{V}$$

and play with it a bit. If you hold the volume constant and increase the temperature, why would the pressure increase? What does that correspond to in physical terms? What would it mean to decrease the volume and keep the pressure constant. How could that be achieved physically? Would the temperature then increase? There is quite a lot to play with in this algebraic formula, which the students may very well see in a semester or two.

- This is an interesting product to look at with the students:

$$(x-1)\left(1+x+x^2+x^3+x^4+\cdots+x^n\right)=x^{n+1}-1$$

  Have them work it out for $n=2$, 3, and 4 and see the pattern. Once they see how it goes, you can derive

$$\left(1+x+x^2+x^3+x^4+\cdots+x^n\right)=\frac{x^{n+1}-1}{x-1}$$

  and use it to estimate such things as $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots$ or the sum of any geometric series.
- This is a good opportunity to review the algebraic rules of commutativity, associativity, etc. Have different sections of the class check different rules for polynomial addition and multiplication. Is it commutative? Associative? What would the distributive law translate to in polynomial addition and multiplication?

▼ Example

A cubic product: $\left(3x^3-2x^2+4x-5\right)\left(x^3-2x^2-4x+1\right)=3x^6-8x^5-4x^4-2x^3-8x^2+24x-5$

▼ Group Work 1: Find the Error

This particular paradox was found carved in a cave-wall by Og the Neanderthal Algebra teacher. Even though it is an old chestnut, it is still wonderful for students to think about, if it is new to them. There is some value in tradition!

Answer: $b-a=0$ and one cannot divide by zero.

▼ Group Work 2: Designing a Cylinder

This activity gives students an opportunity to experiment with an open-ended problem requiring approximations and the use of formulas. Answers will vary.

▼ Homework Problems

Core Exercises: 1, 18, 42, 52, 70, 89, 83

Sample Assignment: 1, 5, 12, 18, 29, 37, 42, 45, 52, 62, 70, 77, 83, 84, 89, 92

GROUP WORK 1, SECTION P.5

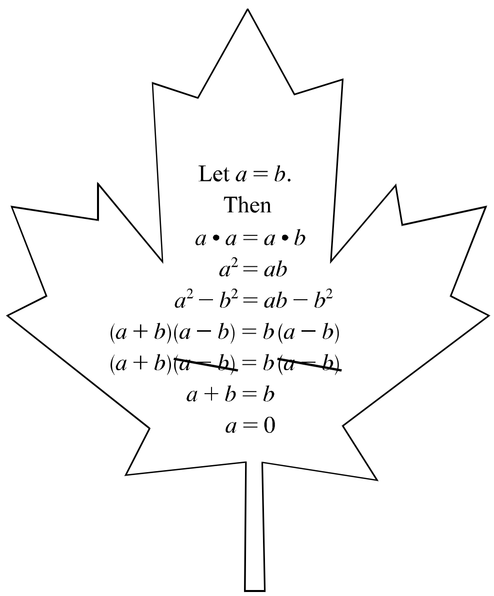
Find the Error

It is a beautiful autumn day. You are collecting leaf samples for your biology class. Your friend Ed is with you, and every time you collect a leaf he says, “School sure is fun, isn’t it?” and you say, “It truly is.” After the tenth time, it starts getting old, but you continue to do it. Suddenly, you smell the sweet smell of sticky lollipop and a voice says, “Plenty of fun if you don’t mind LIES!” You turn around and there is the wild-eyed boy you’ve seen before, with a grin on his face that is even stickier than his lolly.

“What are you talking about?” you ask. “Nobody has lied to me today. In chemistry we learned about chemicals, in music we learned about madrigals, in history we learned about radicals, and in mathematics we learned about. . . ” You pause, trying to think of something that rhymes. And as you pause, he interrupts:

“You learned about LIES. Watch me, old-timer, and learn!”

You wince at being called “old-timer” and you wince again when you realize that the boy has snatched a particularly nice maple leaf from Ed’s hand. Using a felt-tip pen, he writes:



“See that? All those letters and variables bouncing around — it’s a waste of time! Because I’ve just shown that $a = 0$ no matter what you want it to be. Zero. Always zero!”

Is the boy correct? Are all variables equal to zero? That doesn’t seem very “variable” of them.

Save all of algebra! Find the error.

GROUP WORK 2, SECTION P.5

Designing a Cylinder

The volume of a right circular cylinder with height h and radius r is given by $V = \pi r^2 h$.

1. Draw such a cylinder.

2. The equation for the area of a circle is $A = \pi r^2$. Why do you think that "πr^2" also appears in the formula for the volume of a cylinder?

3. Assume you are designing a cylindrical can that has a volume of 120 square inches. Find dimensions of such a can, assuming that it cannot be more than 6 inches tall.

4. Find different possible dimensions for a can with a volume of 120 square inches, again assuming it cannot be more than 6 inches tall.

5. Find dimensions of a can with a volume of 120 square inches that is very long and skinny.

6. Find dimensions of a can with a volume of 120 square inches that is very short and fat.

7. Assume that this is going to be a can of Vegetarian Baked Beans to be manufactured in great quantity and sold to grocery stores. Find the dimensions that would be best to use, in your opinion. Why did you pick those dimensions?

P.6 FACTORING

▼ Suggested Time and Emphasis

$\frac{1}{2}$ – 1 class. Review material.

▼ Points to Stress

1. Factoring expressions by finding common factors.

2. Factoring quadratics by trial and error.

3. Factoring by recognition of special cases.

▼ Sample Questions

- **Text Question:** What does “factoring” mean?
  Answer: Factoring an expression means writing that expression as a product of simpler ones.
- **Drill Question:** Factor the polynomial $12x^3 + 18x^2y$.
  Answer: $6x^2(2x + 3y)$

▼ In-Class Materials

- Factoring expressions ties into two concepts students have seen before: factoring natural numbers, and expanding expressions. Make those connections explicit. Try to get them to see the direct analogy between writing $75 = 3 \cdot 5 \cdot 5$ and writing $x^2y - 4y = y(x+2)(x-2)$. Try to get them to see the inverse relationship between expanding $y(x+2)(x-2) = x^2y - 4y$ and factoring $x^2y - 4y = y(x+2)(x-2)$.
- After doing some more routine examples, show the students that grouping is not always obvious, as in examples like

  $$2xa + 4y + ay + 8x = (2xa + ay) + (4y + 8x) = a(2x + y) + 4(y + 2x) = (2x + y)(a + 4)$$

  Some of the standard formulas can also be awkward. For example, $x^2 - 2$ can be factored, even though we don’t often refer to the number 2 as a “square”.
- Show the students that factoring can be taken to extremes. For example, most people would look at $x^4 + 324$ and think it cannot be factored. (Point out that if the expression were $x^4 - 324$, we could use a formula.). But it can be factored, although the factorization is not obvious:

  $$\begin{aligned} x^4 + 324 &= x^4 + 36x^2 - 36x^2 + 324 = \left(x^2 + 18\right)^2 - 36x^2 \\ &= \left(x^2 + 18\right)^2 - (6x)^2 = \left(x^2 + 6x + 18\right)\left(x^2 - 6x + 18\right) \end{aligned}$$
- It has been proven that any polynomial can be factored into the products of linear factors and irreducible quadratic factors. It has also been proven that if the degree of the polynomial is greater than four there is no formula (analogous to the quadratic formula) that will allow us to find those factors in general.

- This is a good opportunity to remind the students of polynomial division. Have them try to factor an expression such as $x^3 + 2x^2 - 21x + 18$. Now assume you have the hint that $x - 1$ is a factor. Show the students how you can use the hint by dividing the polynomial by $x - 1$ to get a remaining quadratic which is easy to break down.

  Answer: $(x - 3)(x + 6)(x - 1)$

▼ Examples

- A fourth degree polynomial with integer factors:

$$(x - 1)(x - 1)(x + 2)(x - 3) = x^4 - 3x^3 - 3x^2 + 11x - 6$$

- A polynomial that can be factored nicely using the method of Materials for Lecture Point 3:

$$x^4 + 64 = \left(x^2 - 4x + 8\right)\left(x^2 + 4x + 8\right)$$

▼ Group Work: Back and Forth

While expansion and factoring are inverse processes, they are not perfectly symmetrical. It is generally easier to expand than to factor, as illustrated by this activity. Introduce this activity by pointing out that when writing tests, one can't just put up a random polynomial to factor, because the answer might be too hard to find, or too complex to write clearly.

Divide the room into two halves A and B and give each half the corresponding form of the activity. The students expand six expressions and write their answers in the space provided. Emphasize that they should write only their expanded answers, not the work leading up to them, in the blanks. As students finish, they will trade papers, finished As swapping with finished Bs. After the swap, they factor their partner's answer. The pair will then (hopefully) have obtained the original questions back; if not, they should get together and figure out their errors. If a pair finishes early, have them repeat the exercise, making up their own expressions. (If the students are taking a long time, you can have them omit Question 5.)

When closing the activity, note that while it is theoretically possible to do all of them, some (such as Question 5) are extremely difficult to factor.

▼ Homework Problems

Core Exercises: 4, 25, 34, 52, 63, 83, 97, 103, 121

Sample Assignment: 4, 9, 20, 25, 30, 34, 39, 45, 52, 56, 63, 72, 83, 91, 97, 99, 103, 110, 116, 121

GROUP WORK, SECTION P.6

Back and Forth (Form A)

Expand the following expressions. Write your answers (the answers only — do not show your work) at the bottom of this sheet, where indicated.

When you are finished, fold the top of the page backward along the dotted line and trade with someone who has finished Form B. Your job will then be to factor the expressions you receive.

1. $\left(x+\sqrt{3}\right)\left(x-\sqrt{3}\right)$

2. $(x+2)(x+3)$

3. $(2x+3)\left(4x^2-6x+9\right)$

4. $(3r+s)(m+3n)$

5. $(2b-3)^2(x+1)$

Answers:

1.

2.

3.

4.

5.

6.

GROUP WORK, SECTION P.6

Back and Forth (Form B)

Expand the following expressions. Write your answers (the answers only — do not show your work) at the bottom of this sheet, where indicated.

When you are finished, fold the top of the page backward along the dotted line and trade with someone who has finished Form A. Your job will then be to factor the expressions you receive.

1. $(x+3)(x+3)$

2. $(2x-1)(x+2)$

3. $(x-2)\left(x^2+2x+4\right)$

4. $(m+4)^2(n-1)$

5. $(2a+b)(r-5s)$

Answers:

1.

2.

3.

4.

5.

6.

P.7 RATIONAL EXPRESSIONS

▼ Suggested Time and Emphasis

$\frac{1}{2}$ – 1 class. Review material.

▼ Points to Stress

1. Finding the domain of an algebraic expression.

2. Simplifying, adding, subtracting, multiplying, and dividing rational expressions, including compound fractions.

3. Rationalizing numerators and denominators.

▼ Sample Questions

- **Text Question:** What is a rational expression?

  Answer: A rational expression is a fractional expression where both the numerator and denominator are polynomials.

- **Drill Question:** Simplify $\dfrac{(x+2)/(x-3)}{x/(x-2)}$.

  Answer: $\dfrac{(x-2)(x+2)}{x(x-3)}$ or $\dfrac{x^2-4}{x^2-3x}$

▼ In-Class Materials

- One of the most persistent mistakes students make is confusing the following:

$$\begin{aligned} \frac{a+b}{c} &= \frac{a}{c}+\frac{b}{c} \\ \frac{a}{b+c} &= \frac{a}{b}+\frac{a}{c} \quad \longleftarrow \quad \mathfrak{Wrong!} \end{aligned}$$

  Make sure the students understand the difference between these two cases. Perhaps give them the following to work with:

$$\begin{aligned} \frac{u+1}{u} &= 1+\frac{1}{u} \\ \frac{u}{u+1} &= \frac{u}{u+1} \\ \frac{x^3+x+\sqrt[3]{x}}{x} &= x^2+1+x^{-2/3} \\ \frac{x}{x^3+x+\sqrt[3]{x}} &= \frac{x}{x^3+x+\sqrt[3]{x}} \end{aligned}$$

- Ask the students why we like to put rational expressions into standard form by rationalizing the denominator. Note that it depends on the context; often it is a matter of taste. There is nothing inherently "simpler" about $\frac{\sqrt{2}}{2}$ as opposed to $\frac{1}{\sqrt{2}}$ if they are just sitting there as numbers. However, the students would probably prefer to add $\frac{\sqrt{2}}{2}+\frac{\sqrt{3}}{3}+\frac{1}{6}$ than to add $\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\frac{1}{6}$. See if they can come up with other reasons why or instances in which it is convenient to rationalize a denominator.

- This is a good time to start talking about magnitudes. For example, look at $\dfrac{x+6}{x^2+4}$ and ask the question, "What is happening to this fraction when x gets large? What happens when x gets close to zero? What happens when x is large and negative, such as $-1{,}000{,}000$?" The idea is not yet to be rigorous, but to give the students a feel for the idea that a large denominator yields a small fraction, and vice versa. You can pursue this idea with fractions like $\dfrac{x^2+4}{x}$ and $\dfrac{6}{x+(3/x)}$.

▼ Examples

- A compound fraction to simplify: $\dfrac{x+(3/b)}{b+(2x/6)} = 3\dfrac{xb+3}{b\,(3b+x)}$
- A denominator to rationalize: $\dfrac{x}{\sqrt{x}+\sqrt{3}} = \dfrac{x\left(\sqrt{x}-\sqrt{3}\right)}{x-3}$

▼ Group Work: A Preview of Calculus

In the spirit of Exercises 79–84, this activity gives students practice in some of the types of calculations seen in calculus, and foreshadows the concept of a "difference quotient". The students may not all finish. At any point after they are finished with Problem 5(a) you can close the activity by allowing the students to present their answers.

After the answers are presented, if there is student interest, you can point out that in calculus, we want to see what happens when h gets close to 0. Before we have simplified the expressions, if you let $h=0$ all of the expressions become the undefined $\dfrac{0}{0}$. After they are simplified, you obtain expressions in x when you allow $h=0$. Note that technically we cannot allow $h=0$ after we have factored the expressions, because we would have divided by zero; hence one reason for the existence of calculus.

Answers:

1. $\dfrac{\dfrac{1}{x+h}-\dfrac{1}{x}}{h} = -\dfrac{1}{(x+h)\,x}$ **2.** $\dfrac{(x+h)^2-x^2}{h} = 2x+h$ **3.** $\dfrac{\sqrt{x+h}-\sqrt{x}}{h} = \dfrac{1}{\sqrt{x+h}+\sqrt{x}}$

4. (a) $\dfrac{(x+h)^3-x^3}{h}$ (b) $\dfrac{2\,(x+h)-2x}{h}$ (c) $\dfrac{(x+h)^2+(x+h)-\left(x^2+x\right)}{h}$

(d) $\dfrac{5-5}{h}$ (e) $\dfrac{3^{x+h}-3^x}{h}$

5. (a) $3x^2+3xh+h^2$ (b) 2 (c) $2x+h+1$ (d) 0

(e) $3^x\dfrac{3^h-1}{h}$. We cannot simplify this further, nor could we substitute $h=0$ even if we wanted to. You can't win 'em all!

▼ Homework Problems

Core Exercises: 4, 13, 21, 33, 45, 61, 80, 88, 101

Sample Assignment: 2, 4, 7, 10, 13, 18, 21, 29, 33, 38, 45, 54, 61, 68, 74, 80, 88, 95, 101

GROUP WORK, SECTION P.7

A Preview of Calculus

Example 8 in the text involves simplifying an expression similar to this one:

$$\frac{\frac{1}{x+h}-\frac{1}{x}}{h}$$

1. Expressions of this kind occur often in calculus. Without looking at your book, simplify this expression.

2. Now simplify $\frac{(x+h)^2-x^2}{h}$.

3. Now try $\frac{\sqrt{x+h}-\sqrt{x}}{h}$. (**Hint:** Rationalize the numerator, as done in Example 11 in the text.)

These types of expressions are called **difference quotients**. We could write the following abbreviations:

$$DQ\left(\frac{1}{x}\right) = \frac{\frac{1}{x+h}-\frac{1}{x}}{h}$$

$$DQ\left(x^2\right) = \frac{(x+h)^2-x^2}{h}$$

$$DQ\left(\sqrt{x}\right) = \frac{\sqrt{x+h}-\sqrt{x}}{h}$$

In other words, given an expression with x as the variable, we can write its difference quotient. It isn't just a game that algebra teachers play; the difference quotient turns out to be a very important concept in higher math: given information about a quantity, we can figure out how that quantity is changing over time.

4. Let's see if you've picked up on the pattern. Write out the following difference quotients:

(a) $DQ\left(x^3\right)$

(b) $DQ\left(2x\right)$

(c) $DQ\left(x^2+x\right)$

(d) $DQ\left(5\right)$

(e) $DQ\left(3^x\right)$

5. Simplify the difference quotients you found above.

P.8 SOLVING BASIC EQUATIONS

▼ Suggested Time and Emphasis

$\frac{1}{2}$ – 1 class. Essential material.

▼ Points to Stress

1. Solving equations using the techniques of adding constants to both sides of the equation, multiplying both sides of the equation by a constant, and raising both sides of the equation to the same nonzero power.
2. Avoiding the pitfalls of accidentally multiplying or dividing by zero, or introducing extraneous solutions.

▼ Sample Questions

- **Text Question:** Give two different reasons that it is important to check your answer after you have solved an equation such as $2 + \dfrac{5}{x-4} = \dfrac{x+1}{x-4}$.

  Answer: Firstly, we must make sure that our answer is not extraneous, and secondly, we should check that we have not made an error in calculation.
- **Drill Question:** Solve: $(x-2)^2 = 9$.

  Answer: $x = -1$ or $x = 5$

▼ In-Class Materials

- After showing the students how two or three standard linear equations can be solved (such as $4x + 10 = 2$, $2x - 5 = 4$) start introducing other variables such as $4x + 10 = a$ and $4x + a = 2$ and segue to the idea of "solving for x". Once that idea is understood, they should be able to solve $ax + b = c$ for x, by following the pattern of what has been done so far. Then show them how to solve it for b, and ask them to solve it for c. If they understand what the phrase means, the response should be that the equation has already been solved for c, or that we should write $c = ax + b$.
- There are two main pathologies that students may encounter when solving linear equations: $0 = 0$ and $1 = 0$. While it is not a good idea to dwell on the pathologies, as opposed to spending time on the cases where the students will spend a majority of their time, it is good to address them, because they *do* come up, in math class and in real life. Have them attempt to solve the following equations, and check their work:

$$\begin{aligned} 3x + 4 &= x + 6 \\ 3x + 4 &= x + 6 + 2x - 2 \\ 3x + 4 &= x + 6 + 2x + 2 \end{aligned}$$

  By the time this example is presented, most of the students should be able to obtain $x = 1$ for the first equation, and to check this answer. The second one gives $0 = 0$. Point out that $0 = 0$ is always a true statement. (If you like, introduce the term "tautology".) It is a true statement if $x = 1$, it is a true statement if $x = 2$, it is a true statement if the author George R. R. Martin writes another book, it is a true statement if he does not write another book. So the students can test $x = 1$, $x = 2$, $x = -\sqrt{2}$ in the second equation, and all of them will yield truth. The third statement gives $1 = 0$ (actually, it gives $0 = 4$, but we can multiply both sides of the equation by $\frac{1}{4}$). This is a false statement, no matter what value we assign to x. Point out "Currently, it is false to say that this piece of chalk is blue. Now, is there a value I can assign to

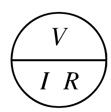
x to change that?" Again, test various values of x in the third equation to see that, no matter what, we get that the LHS is always 4 less than the RHS.

- If the equation $PV = nRT$ was introduced earlier (where P is pressure, V is volume, n is quantity of gas in moles, T is temperature, and R is a constant called the ideal gas constant) it is a good time to bring it up again. Ask how we could experimentally determine the value of R. (Answer: Measure P, V, n, and T, and solve the equation for R.) Ask the students to solve the equation for T and then for P.

- Modern electrical engineering students learn Kirchoff's Voltage Law: $V = IR$. (Voltage equals current times resistance.) In the 1950s, some employees of the U.S. Navy taught their electrical engineers "Kirchoff's Three Voltage Laws":

$$\begin{aligned} V &= IR \\ \frac{V}{I} &= R \\ \frac{V}{R} &= I \end{aligned}$$

  They were given this mnemonic diagram to help them remember the three laws:



  If you want to know, say, Resistance, you cross out the R and are left with $\frac{V}{I}$. Ask the students if modern students are being cheated by only learning about one of these laws, and not learning the diagram. Hopefully they will get the idea that you can obtain the three laws by solving for whichever variable you need.

▼ Example

An equation with an extraneous solution:

$$\frac{x^2}{x+1} = \frac{x+2}{x+1}$$

This simplifies to $x^2 - x - 2 = 0$, either by multiplying by $x + 1$ directly or by first subtracting the RHS from the LHS and simplifying. The students don't yet know how to solve quadratic equations, but they can verify that $x = -1$ and $x = 2$ are solutions. $x = -1$ is extraneous.

▼ Group Work 1: Leaving the Nest

This lightly foreshadows the next section, in that students are asked to use an equation to solve a real-world problem. This problem can be modified to make the original savings a parameter instead of the constant 100. If students solve Problems 7 and 8 in different ways, make sure that the class sees the various methods used.

Answers:

1. $\frac{1100 \cdot 2 - 300}{70} \approx 27.14$. We round up to 28 months.

2. $\frac{1900}{x}$ $\left(\text{technically } \left\lceil \frac{1900}{x} \right\rceil\right)$

3. $3 = \frac{1900}{x} \quad \Rightarrow \quad x = \$633.33$ per month

4. $\frac{2200/2 - 300}{70} \approx 11.43$; we round up to 12 months

5. $\frac{2200/3 - 300}{70} \approx 6.19$; 7 months

6. $\frac{2200/(n+1) - 300}{70}$ months

7. $\frac{2200/(n+1) - 300}{70} = 3$, $n \approx 3.31$; 4 roommates

8. $\frac{2200/(n+1) - 300}{70} = \frac{1}{4}$, $n \approx 5.93$; 6 roommates

▼ Group Work 2: The Mathematics of Pizza

The first question isn't an algebra question, but a warm up to make sure that the students remember the formula for the area of a circle (it is also a question that comes up in dorm rooms quite frequently.) If you want to use this as an opportunity to review significant figures, you may want to be strict that all answers should be given to (say) four significant figures.

Answers:

1. The large pizza is larger than two small pizzas—49π in^2 versus 32π in^2.

2. $\pi r^2 = 100 \Rightarrow r = \sqrt{\frac{100}{\pi}} \approx 5.642$ in. The diameter is about 11.28 in.

3. $d = 2\sqrt{\frac{n^2}{\pi}} \approx 1.128n$

4. $n^2 = 49\pi \Rightarrow n \approx 12.41$

▼ Homework Problems

Core Exercises: 4, 9, 27, 47, 68, 84, 100, 105

Sample Assignment: 4, 5, 9, 13, 20, 27, 32, 39, 47, 50, 62, 68, 74, 84, 93, 100, 102, 105

GROUP WORK 1, SECTION P.8

Leaving the Nest

1. I am a mathematics professor who lives in a town where an inexpensive apartment costs $\$1100$ per month. In order to rent an apartment here, I need the first month's rent up front, and a security deposit equal to the monthly rent. Assume that I have $\$300$ saved up in the bank, and I can save $\$70$ per month from my professorial salary. How long will it be before I can move out of my parent's basement?

2. Depressing thought, eh? Well, one way to move out sooner would be to save more money. How long will it be before I can move out, assuming I can save x dollars per month?

3. How much would I have to save per month if I wanted to move out in three months?

4. Realistically, I can't really save all that much more money. Math supplies do not come cheap. Let's go back to the situation where I am saving $\$70$ per month. Another way I could move out sooner would be to get a roommate. How long would it be before I could move out, assuming a roommate would pay half of the first month's rent and the security deposit?

5. How long would it take if I had 2 roommates?

6. How about n roommates?

7. How many roommates would I have to have if I wanted to move out in three months?

8. For reasons best left to your imagination, it would make things a lot easier if I moved out next week. How many roommates will I need for this to happen?

GROUP WORK 2, SECTION P.8

The Mathematics of Pizza

1. A small pizza from Raineri's Pizza is 8 inches in diameter. A large pizza is 14 inches in diameter, and costs twice as much. Are you better off buying two small pizzas or one large one, assuming you want as much pizza as possible?

2. Pizzas aren't always round. In Chicago, for example, pizza often comes in rectangles. Assume you make a pizza in a ten-inch by ten-inch pan. If you wanted a round pan that would give you the same amount of pizza, what would the diameter of that round pan be?

3. What would be the radius of a round pan that gave you as much pizza as an $n \times n$ square pizza?

4. How big a square pan would you need to give you as much pizza as a large pizza from Raineri's Pizza?

P.9 MODELING WITH EQUATIONS

▼ Suggested Time and Emphasis

1 class. Essential material.

▼ Points to Stress

1. Solving applied problems described verbally.
2. Presenting the solution process in a clear, organized way.

▼ Sample Questions

- **Text Question:** Discuss one of the problems presented in this section. You don't have to know all the specific numbers, but describe the problem and the method of solution.
- **Drill Question:** A car rental company charges \$20 a day and 30 cents per mile for renting a car. Janice rents a car for three days and her bill comes to \$72. How many miles did she drive?
  Answer: The total cost is $C = 20d + 0.30m$, where d is the number of days and m is the mileage. In this case we have $72 = 20(3) + 0.30m \Rightarrow m = 40$, so Janice drove 40 miles.

▼ In-Class Materials

- Some college algebra students will have seen this material at one point in their lives, and may have many different ideas and habits. It is important to make your requirements explicit: is it acceptable to present an answer with no work at all? With work shown but barely decipherable? Do you require the students to explicitly go through the steps described in the text? Do you require them to write out a table (as in the examples in the text)? Much conflict can be avoided if your requirements are specified in a handout at the outset.
- Here is a simple-sounding problem: A person is currently making \$50,000 a year (after taxes) working at a large company and has an opportunity to quit the job and make \$80,000 independently. Is this a good deal financially? Elicit other considerations such as: Insurance tends to cost quite a lot for an individual — the company is no longer paying for that. Taxes will be roughly 30%. The large company probably pays some percentage (say 2%) toward a retirement account. Discuss this situation with your class, trying to come up with a linear equation that converts an annual gross income for an independent contractor to a net income, and then figure out how much the independent contractor would have to charge in order to net \$50,000. Other hard-to-quantify considerations may come up such as paid vacations, sick leave, "being one's own boss", the tax benefits of having a home office, quality of life, and so forth. Acknowledge these considerations, and perhaps (for purposes of comparison) try to quantify them. Perhaps assume that the tax benefits for the home office add 5% to the gross income. Ask the class if the pleasure of not having a boss is worth, say, \$2000 a year. (On the other hand, would they would get any work done at all without a boss standing over them?) This is a real-life modeling situation that, every year, more and more people find themselves thinking about.
- This section is particularly suited to having students come up with their own problems to solve, or have others solve. Once they understand the concept, with a little (or a lot of) thought, they should be able to come up with practical, relevant problems. If students need a little prompting, you can wonder how many

2 GB flash drives one would have to buy to store x full-length movies, how many people one could invite to a party that has a firm budget of $\$200$ (have them list fixed costs and per-person costs), how many miles one could drive a car (paying all expenses) for $\$3000/$year, etc.

- Point out that not all behaviors are best modeled linearly. For example, assume that the temperature of a thermos of coffee is $160°$F, and after a half hour it is $120°$. Show how we can write a linear model: $T = 160 - 80t$. Show that, according to the model, the coffee will reach $0°$ in two hours and will freeze solid a little less than a half hour after that! A much better model of coffee cooling is be the exponential model $70 + 90\,(0.605)^t$, assuming that the temperature of the room is $70°$.

▼ Examples

- A straightforward mixture problem: A child makes two quarts of chocolate milk consisting of 30% syrup and 70% milk. It is far too sweet. How much milk would you have to add to get a mixture that is 5% syrup?
  Answer: 10 quarts
- A straightforward job-sharing problem: If it takes Mike two hours to mow the lawn, and Al three hours to mow the lawn, how long does it take the two of them, working together?
  Answer: 1.2 hours
- A classic (i.e. old) trick question: If it takes two people three hours to dig a hole, how long does it take one person to dig half a hole?
  Answer: You can't dig half a hole. Don't worry; my students did not laugh either.

▼ Group Work 1: How Many People?

This modeling problem has a lot of data, and is designed to teach students to distinguish between fixed and variable costs. As in the real world, some data are irrelevant (e.g. the maximum capacity of the hall) and some seem irrelevant, but are not (e.g. the length of the reception).

If a group finishes early, ask them how many people could be invited if the budget is $\$3000$ and 70% of the people who have been invited attend. (Answer: 163)

Answers:

1. $\$3799.50$ (rounding to $\$3800$ is permissible)
2. Fixed costs are $F(x) = 2000 + 150 \cdot 3 + 400$ and variable costs are $V(x) = \left(15 + \frac{9}{3} + 1.11 \cdot 0.9\right) x$. Therefore, $C(x) = F(x) + V(x) = 18.999x + 2850$.
3. 60. If a student answers 60.558, they should include an explanation of what they mean by that, given that 0.558 of a person doesn't make sense on the face of it. Note that if a student rounds up to 61, they may go over budget. Given that there are approximations involved, it would be better to err on the side of caution, perhaps hosting 55 people rather than pushing it to the limit of 60.

▼ Group Work 2: Some Like it Hot

This is a straightforward mixture problem.

Answers:

1. $\dfrac{25}{10{,}000} = \dfrac{1}{400}$ gallon
2. $\dfrac{1}{100} = \dfrac{\frac{1}{800} + x}{\frac{1}{2} + x} \Leftrightarrow x = \dfrac{1}{264}$ gallon

▼ Group Work 3: Which Gas is Cheaper?

The numbers in this problem were not made up; they were true in Cedar Falls, Iowa as of August 2014. The amount of ethanol use, and whether or not consumers have a choice, varies from state to state. Feel free to change the numbers around based on your research, your state, and your car.

Answers:

1. **(a)** $\$13.80$ **(b)** $\$13.39$ **(c)** Non-ethanol gas
2. **(a)** 41.4 cents. (Notice that the distance we are traveling turns out to be irrelevant.) **(b)** Ethanol gas **(c)** Non-ethanol gas
3. **(a)** 2.17 miles per gallon **(b)** Non-ethanol gas **(c)** Ethanol gas
4. Cost $= \dfrac{A + c}{B + m} d$

▼ Homework Problems

Core Exercises: 6, 20, 23, 41, 51, 64

Sample Assignment: 2, 6, 9, 12, 20, 23, 29, 33, 41, 45, 51, 58, 64

GROUP WORK 1, SECTION P.9

How Many People?

Jessica and Trogdor are getting married! They are planning the reception, and have come up with the following data:

- The reception hall costs $2000 to rent.
- The reception hall holds a maximum of 500 people.
- The reception will last three hours.
- Each person's dinner will cost $15.
- On average, every person that drinks alcohol will cost an extra $9.
- About one-third of the attendees will drink alcohol.
- Each piece of wedding cake will cost $1.11.
- 90% of the guests will eat a piece of wedding cake.
- The DJ charges $150 per hour.
- The decorations (flowers, streamers, groom bobbleheads, etc.) are going to cost $400.

Answer the following questions:

1. How much would the reception cost if 50 people attended?

2. Write an algebraic formula describing the cost of x people attending their reception.

3. How many people could come if Jessica and Trogdor have a total of $4000 to spend?

GROUP WORK 2, SECTION P.9

Some Like it Hot

I like my chili spicier than my sister does. I like mine to be 1% habañero sauce, and she likes it to be 0.25% habañero sauce. Here is the plan: I'm going to make a gallon of chili the way my sister likes it. I'll measure out a half-gallon, and put it in a container for her. Then I will add some more habañero sauce so that the remaining chili will be the way I like it.

1. How much habañero sauce should I put in initially?

2. How much extra habañero sauce should I put in my portion after I've divided it up?

GROUP WORK 3, SECTION P.9

Which Gas is Cheaper?

As of 2014, when you buy gas in Iowa, you have two choices. Gas with ethanol, which is cheaper, or gas without, which is more expensive. Whenever I fill up the family car with gasoline, I use the cheaper gas, for three reasons:

1. It is cheaper.
2. Ummmm...
3. Kittens?

When my wife Laurel fills up the car, she uses the more expensive gas. Her reasoning is that the car gets worse gas mileage with the ethanol gas, so we have to burn more of it to go the same distance.

Being scientifically-minded people, we decided to figure out which of us was right. She did all the work of figuring out our average gas mileage with each kind of gas. I did all the work of going to a gas station and writing down how much each kind cost with my pencil. Here are our results from August 2014:

With ethanol: Cost $\$3.45$ per gallon; mileage 25 miles per gallon.

Without ethanol: Cost $\$3.75$ per gallon; mileage 28 miles per gallon

1. Assume we drive 100 miles.

(a) How much would it cost to drive 100 miles with the ethanol gas?

(b) How much would it cost to drive 100 miles with the non-ethanol gas?

(c) Which gas should we buy in this case?

Notice that right now, the price difference between the two types of gas is 30 cents. This price differential changes from year to year. For example, two years ago, it was closer to a dime. We now want to figure out how big (or small) the price difference would have to be in order for the answer to Problem 1(c) to change.

2. Assume that the ethanol gas is still $\$3.45$ per gallon, but now the difference in price between gas without ethanol and gas with ethanol is c cents.

(a) What value of c makes the cost of driving 100 miles the same regardless of the type of gas used?

(b) Which gas should we pick if c were larger than the value you just computed?

(c) Which gas should we pick if it were smaller?

A crucial piece of this puzzle is the difference in gas mileage. The numbers I gave you (25 miles per gallon and 28 miles per gallon) are based on our car. I am going to enjoy typing this next sentence...
Your mileage may vary.

3. Assume that ethanol gas costs $\$3.45$ per gallon, and non-ethanol gas costs $\$3.75$ per gallon, and that your car gets 25 miles per gallon with ethanol gas. Let m be the improvement (in mpg) with non-ethanol gas.

(a) What value of m makes the cost of driving 100 miles the same regardless of the type of gas used?

(b) Which gas would you pick if m was larger than the value you just computed?

(c) Which gas would you pick if it were smaller?

4. Now let's come up with a general model. Assume that A is the cost of ethanol gasoline, c is the difference in cost between ethanol and non-ethanol gasoline, B is the mileage of a car using ethanol gasoline, and m is the difference in mileage between ethanol and non-ethanol gasoline. Write a model that gives the cost to drive d miles using non-ethanol gasoline as a function of A, c, B, and m.